INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 15, 1407–1415 (1992)

THE EIGENVALUE SPECTRUM OF THE RAYLEIGH EQUATION FOR A PLANE SHEAR LAYER

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ANALYSIS

The eigenvalue spectrum of the Rayleigh equation[1] is examined in this paper using three different solution techniques. The Rayleigh equation governs the inviscid instability of linear disturbances superimposed on a parallel basic flow $U(y)$. The equation may be written,

$$\left[(\alpha U-\omega)\left(\frac{d^2}{dy^2}-\alpha^2\right)-\alpha\frac{d^2U}{dy^2}\right]\hat{v}=0, \tag{1}$$

where $\hat{v}$ is the complex amplitude of the velocity perturbation in the y-direction. α denotes the wave number and ω is the frequency. In a spatial stability analysis it is assumed that the frequency is real and the wave number is complex. Thus, the disturbances are periodic in time and grow or decay with downstream distance. Since the homogeneous free shear layer is convectively unstable, these solutions are related to the asymptotic response of the shear layer to a periodic initial disturbance.

For the free shear layer the boundary conditions may be written,

$$\hat{v}\to 0, \qquad y\to\pm\infty. \tag{2}$$

Equations (1) and (2) provide a homogeneous boundary value problem that may be solved for the dispersion relationship,

$$\alpha=\alpha(\omega). \tag{3}$$

In the present analysis, a simple second-order finite difference scheme and two spectral methods are used to discretize the Rayleigh equation. These include the Chebyshev tau (CT) and a Chebyshev collocation (CC) method. Each of the three approximation methods produces an eigenvalue problem in which the eigenvalue, α, appears non-linearly. That is,

$$D_3(\alpha)\mathbf{v}=0, \tag{4}$$

0271–2091/92/241407–09$09.50

Received 7 August 1991
Revised 27 April 1992

where

$$D_3(\alpha) = \mathbf{C}_0\alpha^3 + \mathbf{C}_1\alpha^2 + \mathbf{C}_2\alpha + \mathbf{C}_3.$$

$\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2$ and $\mathbf{C}_3$ are the $(N-1) \times (N-1)$ coefficient matrices of the lambda matrix $D_3(\alpha)$. N denotes either the number of grid points in the FD and CC methods or the number of the Chebyshev polynomials used in the CT method. The components of the eigenvector, $\mathbf{v}$, are either the expansion coefficients of the Chebyshev series approximation or the solution vectors themselves. The eigenvalues of the system are the roots of the characteristic equation,

$$\det|D_3(\alpha)| = 0. \tag{5}$$

Two methods have been used to solve the eigenvalue problem (5). In the first, the linear companion matrix (LCM) method, the problem is reduced to a generalized eigenvalue problem through the introduction of a new solution vector. The resulting companion matrix of the matrix polynomial is of order $3 \times (N-1)$ in the present case. This method provides an approximation to the entire eigenvalue spectrum. The second technique used, the matrix factorization (MF) method, involves only matrices of order $(N-1)$. This method, however, resolves only a subset of the entire eigenvalue spectrum. The details of the formulation of the two methods can be found in Bridges and Morris.[2]

The eigenvalue spectrum of the Rayleigh equation consists of a discrete component and two continuous parts. The discrete component of the spectrum is associated with the convective instability. Consequently, when one is concerned only with a criterion for instability, for a particular shear flow, the continuous part of the eigenvalue spectrum is generally ignored.[3] Nevertheless, an eigenvalue spectrum is made complete only with the inclusion of the continuous branches. An arbitrary disturbance cannot be represented properly without knowledge of the complete eigenvalue spectrum. Therefore, a good approximation to the entire eigenvalue spectrum is important to the solution of the initial-value problem in hydrodynamic stability.

In addition to a finite number of discrete values of α that satisfy the dispersion relation (3), there are two branches of a continuous eigenvalue spectrum associated with the critical point singularity at $y=y_c$, of the Rayleigh equation. That is, at

$$\alpha U(y_c) - \omega = 0. \tag{6}$$

In the present calculations, the basic velocity profile for the free shear layer is assumed to be,

$$U(y) = \tfrac{1}{2}[1 + \tanh(y)]. \tag{7}$$

In this case, the two branches of the continuous spectrum are given by

$$\text{(i)} \quad \alpha_i = 0; \qquad \alpha_r \geqslant \omega, \tag{8}$$

$$\text{(ii)} \quad \alpha_i \in R; \qquad \alpha_r \to \infty. \tag{9}$$

Case[4] considered this branch of the eigenvalue spectrum for Couette flow. The eigenfunctions decay exponentially away from $y=y_c$ and their slope is discontinuous at $y=y_c$. Equation (9) represents the special case where $U(y_c)=0$. The two other branches of the continuous spectrum are related to bounded solutions of the asymptotic form of the Rayleigh equation in the far field,

$$\frac{d^2}{dy^2}\hat{v} - \alpha^2\hat{v} = 0, \tag{10}$$

since $d^2U/dy^2 \to 0$ as $y \to \pm\infty$. These are the components of the continuous spectrum associated with the domain unboundedness and can be combined as,

$$\alpha_r = 0, \qquad \alpha_i \in R. \tag{11}$$

The corresponding eigenfunctions are purely periodic.

In order to distinguish more easily between the discrete component and the continuous component of the eigenvalue spectrum, a transformation of the eigenvalue has been used in conjunction with the matrix factorization method. The transformation is given by

$$\hat{\alpha} = \frac{1}{(\alpha_f - \alpha)}. \tag{12}$$

The lambda matrix then becomes,

$$\hat{D}_3(\hat{\alpha}) = \hat{\mathbf{C}}_0\hat{\alpha}^3 + \hat{\mathbf{C}}_1\hat{\alpha}^2 + \hat{\mathbf{C}}_2\hat{\alpha} + \hat{\mathbf{C}}_3. \tag{13}$$

This transformation insures that the eigenvalues of $D_3(\alpha)$, that are in the vicinity of α_f, appear in the set of eigenvalues of the dominant solvent of $\hat{D}_3(\hat{\alpha})$. A solvent of $\hat{D}_3(\hat{\alpha})$ is said to be dominant if the magnitudes of every one of the eigenvalues of the solvent are greater than the magnitudes of the eigenvalues of the other solvents of $\hat{D}_3(\hat{\alpha})$. An algorithm developed by Dennis *et al.*[5] has been used to find the dominant solvent of the matrix polynomial. The eigenvalues of the dominant solvent can then be obtained using standard techniques for algebraic eigenvalue problems. The eigenvalues calculated by the three spatial discretizations are refined with the iterative procedure of Lancaster.[6]

The application of spectral and finite difference approximations, in conjunction with global eigenvalue solution methods, can be found in Bridges and Morris[2] and Malik,[7] among others. Further details of the present analysis can be found in Liou.[8]

RESULTS

A square-root transformation is used to map the unbounded physical domain to the domain, $[-1, 1]$, on which the Chebyshev polynomials are defined. The transformation used is,

$$z = \frac{y}{(r^2 + y^2)^{1/2}}, \tag{14}$$

where r is a scaling factor. The transformation produces no singularities at the end points of the transformed domain and the convergence of the global approximations is retained.[9] The scaling factor r controls the distribution of grid points. Its optimum value, for which the solutions are most accurate, may depend on both the number of grid points and the discretization scheme. Boyd[10] used a steepest descent method to calculate the optimum choice of the mapping parameter in the application of a Chebyshev polynomial approximation to a known, explicit function. However, analytic evaluation of the optimum mapping parameter for boundary value problems would be extremely difficult. In the present calculations the optimum value of r has been determined experimentally. In addition, Boyd[11] used mappings in the complex plane to avoid singularities associated with the critical point of the Rayleigh equation and its branch cut. Optimum mapping parameters were proposed. In the present calculations the discrete eigenvalues represent unstable solutions and no contour deformation is required to avoid the critical point singularity.

Figure 1 shows the order of accuracy of the predicted discrete spectrum using the LCM method and for $\omega = 0{\cdot}2$. The absolute error, ε, is defined by,

$$\varepsilon = |\alpha - \alpha_s|, \tag{15}$$

where α_s is the eigenvalue calculated using a shooting method. In the shooting procedure, the Rayleigh equation is integrated in the interval, $[-7, 7]$, with 8000 grid points using a fourth-order, fixed step size Runge–Kutta method. This gives,

$$\alpha_s = 0{\cdot}3826245 - i0{\cdot}2276903. \tag{16}$$

For the fine grid spacing, the value of α_s can be considered *numerically* exact and may be used to compare with solutions of other numerical methods. For each of the three spatial discretization methods, the predicted discrete eigenvalues agree very well with α_s, even for small values of N. In fact, for $N = 10$, the error for each of the three discretization methods are approximately 1%. Nevertheless, the spectral methods converge faster than the FD method. In Reference 8 it has also been shown that the dependence of the prediction of the discrete eigenvalues on the value of the scaling factor r is weak, and this dependence diminishes as the value of N increases.

It should be noted that, for each of the three discretization methods, the predicted discrete eigenvalue spectrum obtained with the LCM method match at least to the eighth digit with those obtained by the MF method with $\alpha_f = \alpha_s$. The effect of the value of α_f on the MF predictions of the discrete eigenvalue is also minimal. For instance, for $N = 28$ and $\alpha_f = (-0{\cdot}3, 0{\cdot}2)$, the magnitude of the error of the CC/MF prediction of the discrete eigenvalue is 0·0050, compared to 0·0036 for the prediction with the CC/LCM combination. Thus, for spatial linear instability calculations that

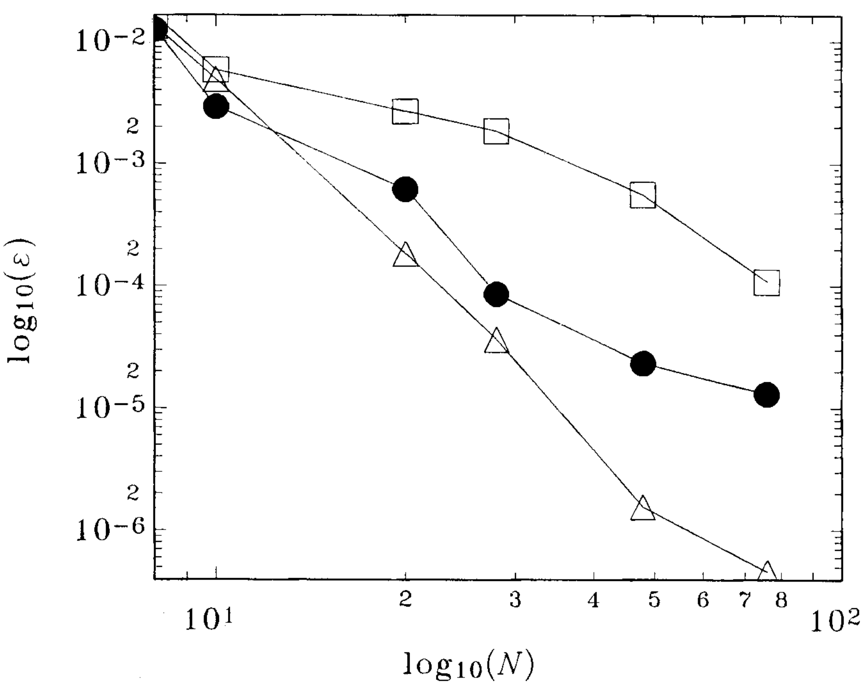


Figure 1. Absolute Error ε, equation (15), in the discrete eigenvalue spectrum: -△-, Chebyshev collocation method; -□-, finite difference method; -●-, Chebyshev tau method

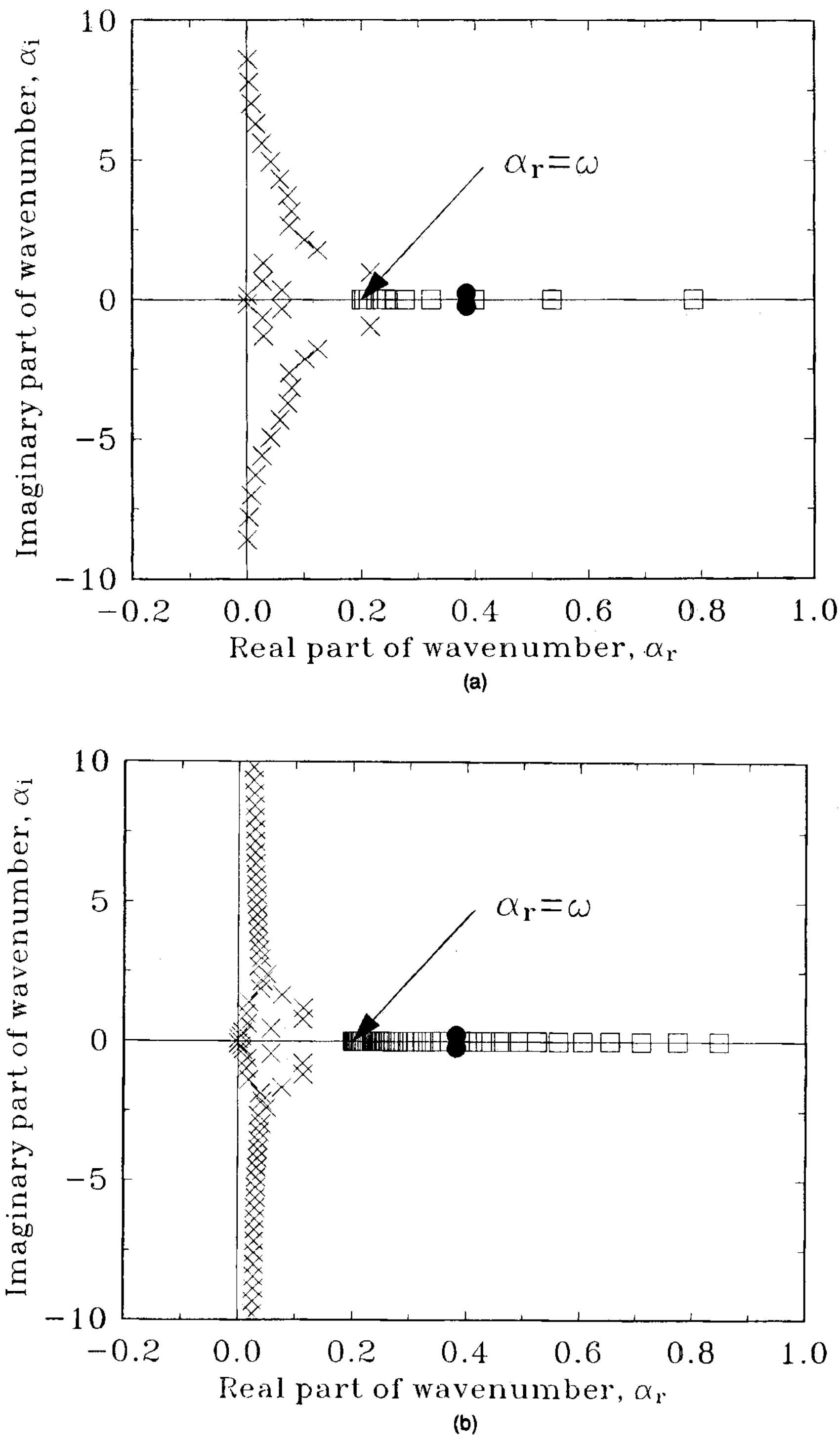


Figure 2. Eigenvalue spectra for $\omega = 0{\cdot}2$: ×, eigenvalues associated with the unbounded domain; □, eigenvalues associated with the singularity of the Rayleigh equation; ●, discrete spectrum. (a) $N=17$; (b) $N=76$

require the discrete eigenvalues, either one of the eigenvalue solution techniques, the LCM and the MF method, can predict the discrete part of the eigenvalue spectrum satisfactorily.

Figures 2(a) and 2(b) show the entire calculated eigenvalue spectrum for $\omega = 0{\cdot}2$ and $N = 17$ and $N = 76$, respectively. The finite difference discretization is used and the LCM method is used to obtain the eigenvalue spectrum. The approximation to the continuous spectra associated with equations (8) and (11) can be identified clearly. The branches described by equation (9) has not been shown due to the large magnitude of the eigenvalues. As can be observed, the resolution of both components of the continuous spectra improves with increases in the value of N. It is to be expected that, as $N \to \infty$, the approximate spectra would converge to the analytic expressions.

Figures 2(a) and 2(b) also show how the presence of the continuous eigenvalue spectra can conceal or mask discrete eigenvalues in the wave number plane. However, in the phase velocity plane, the continuous spectra are separated from the discrete spectrum. This is shown in Figure 3. The complex phase velocity, $c = \omega/\alpha$, for the discrete eigenvalue, for $\omega = 0{\cdot}2$, is $(0{\cdot}51845, -0{\cdot}87404)$. Thus, in the present case, the discrete spectrum can be observed better in the phase velocity plane.

Figures 4(a)–4(c) show the eigenvalue spectra for $\omega = 0{\cdot}2$ using the CC, the FD and the CT methods, respectively. The discrete eigenvalues are predicted accurately for all the discretization methods. The characteristics of the continuous spectrum, associated with the critical point, given by equation (8), are captured by all the discretization methods. The accuracy of the numerical predictions also increases with increasing values of N. Despite the oscillatory nature of the corresponding eigenfunctions as $y \to \infty$, the CC and FD methods give good approximations to the continuous spectrum associated with the domain unboundedness. Again, the accuracy increases with increasing values of N. However, the CT predictions of this component of the

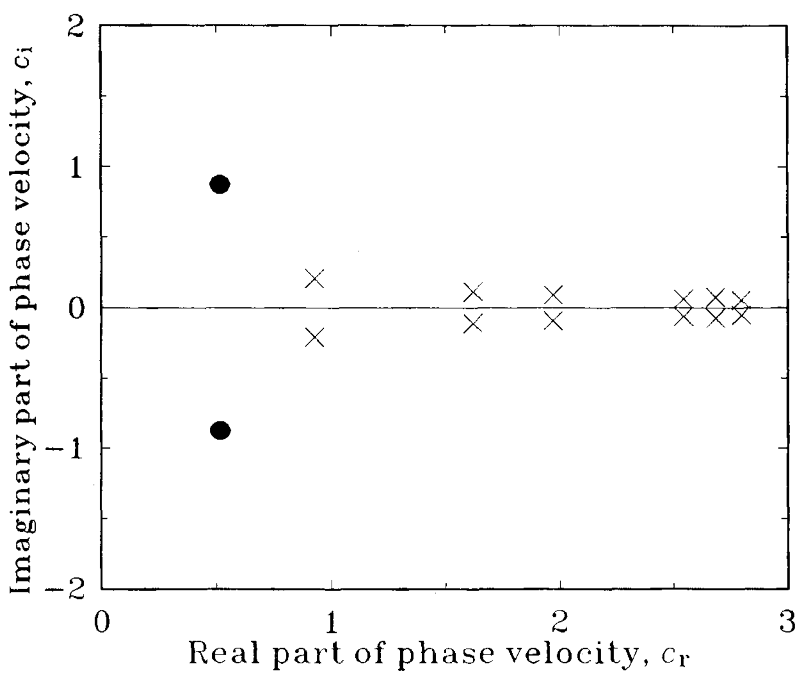
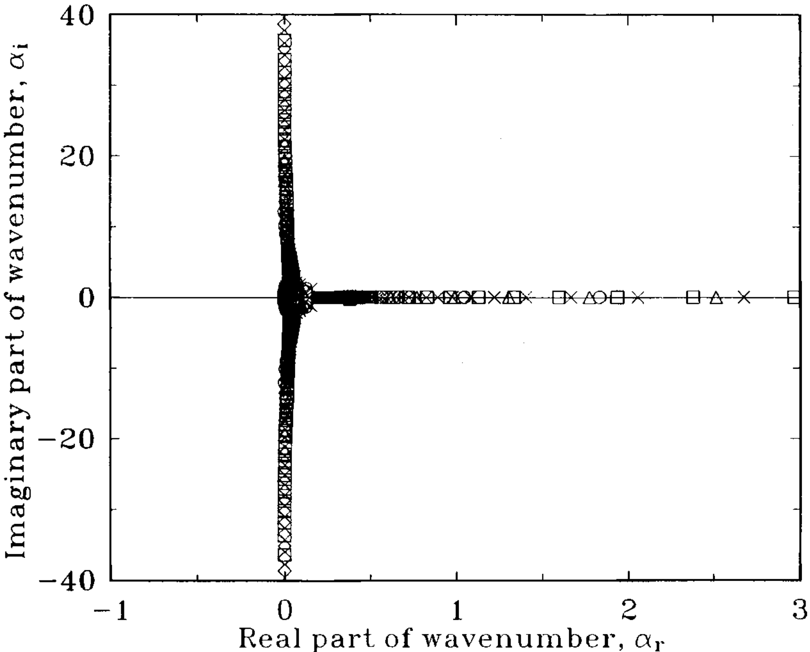
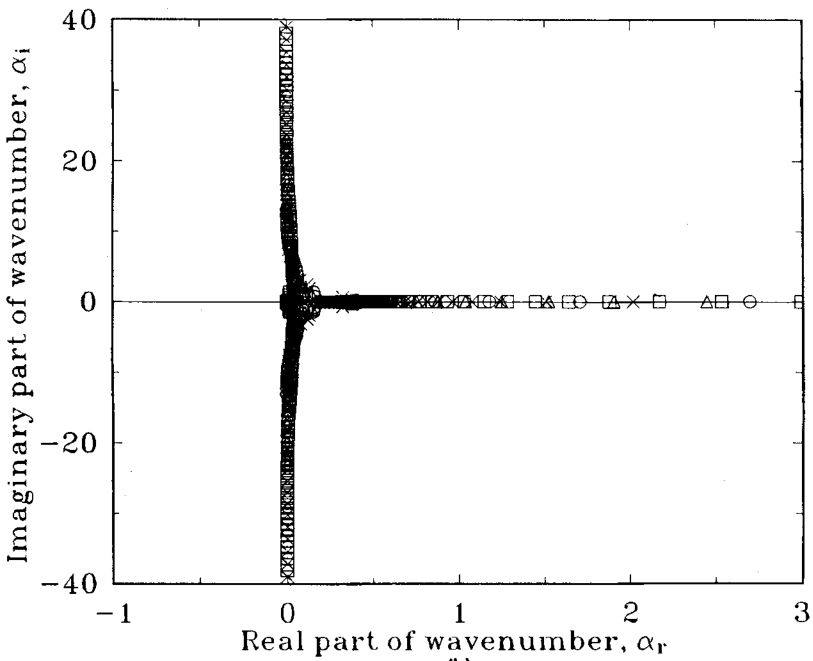


Figure 3. Eigenvalue spectra in the complex phase velocity plane. $\omega = 0{\cdot}2$, $N = 17$. Legend: See Figure 2

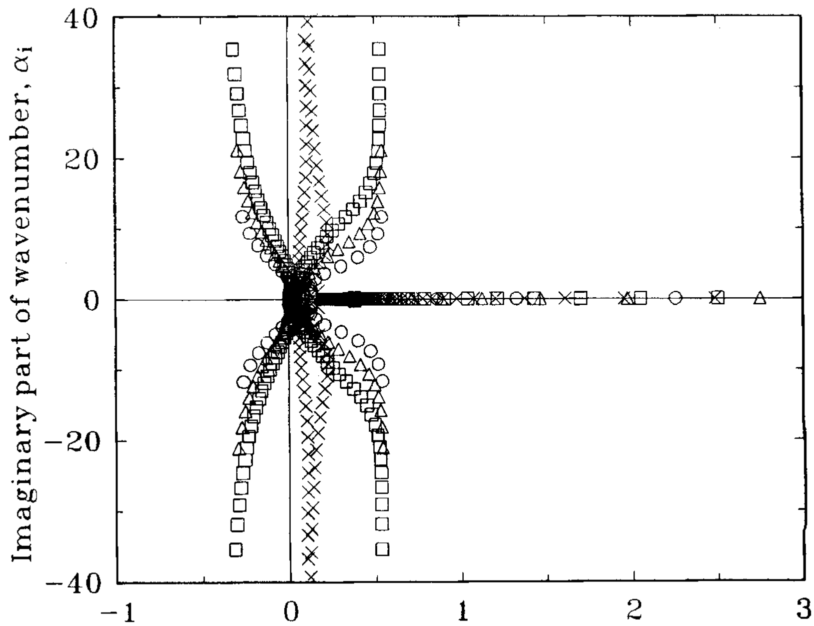


(a)



(b)

Figure 4. Eigenvalue spectra for $\omega=0{\cdot}2$: ○, $N=26$, $r=2{\cdot}0$; △, $N=46$, $r=2{\cdot}0$; □, $N=76$, $r=2{\cdot}0$; ×, $N=76$, $r=0{\cdot}5$. (a) Chebyshev collocation method; (b) Finite difference method; (c) Chebyshev tau method



(c)

Figure 4. (Continued)

continuous spectrum are sensitive to the value of the mapping parameter r, even for relatively high values of N. It is possible that this weakness in the CT predictions is due to their inability to approximate the rapidly oscillating eigenfunctions in the mapped domain. However, we have no satisfactory evidence at this time to support this proposition.

CONCLUSIONS

Three discretization schemes, including a second-order finite difference, a Chebyshev tau and a Chebyshev collocation method, have been applied to calculate the eigenvalue spectrum that describes the spatial inviscid instability of a free-mixing layer.

All of the discretization methods are capable of predicting the discrete spectrum as well as the continuous spectrum associated with the critical point singularity of the Rayleigh equation. The continuous spectrum associated with the unbounded domain can also be predicted well by the three methods. Nevertheless, the Chebyshev tau predictions of this latter branch of the eigenvalue spectrum are somewhat sensitive to the value of the mapping parameter in the square-root transformation. It appears that the square-root transformation used here is a viable alternative to the three transformations tested by Grosch and Orszag[9] for free shear flows. The global eigensolution methods studied here may be applied very efficiently to obtain either an approximation to the complete eigenvalue spectrum or initial guesses for a local shooting procedure for the discrete part of the spectrum.

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